Astroparticle Physics
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Write your name and S number on every sheet
You don't have to use separate sheets for every question.
There are **6 questions** with a total number of marks: 37

WRITE CLEARLY

(1) (Total 8 marks)
In the framework of the first-order Fermi mechanism, the acceleration process of charged particles is carried out in shock fronts which occur, for example, in supernovae.

(a) (1 mark)
Suppose that the energy gain per cycle is $\Delta E = \epsilon E_0$, where E_0 is the particle initial energy. What is the energy of the particle after it has passed the cycle n times?

(b) (1 mark)
The probability that a particle is confined in the accelerator region is P_c per cycle. How many of N_0 initial particles have still remained in the acceleration process after n cycles?

(c) (2 marks)
Determine the energy spectrum $\frac{dN}{dE}$ of all particles that have undergone the acceleration process, depending on the initial energy E_0.

(d) (1 mark)
Typical velocities of shock fronts from supernovae are $v_S \simeq 6\ 10^4$ km/s. Compute the spectral index of the energy spectrum if the probability that a particle remains confined is $P_c = 70\%$.

(e) (1 mark)
For a particle of initial energy $E_0 = 1$ MeV, how many times does it have to go through the shock front until it reaches an energy of 1 TeV ?

(f) (2 marks)
Demonstrate that the Larmor radius r_L of a particle of charge Ze and energy E in a magnetic field of strength B is given by $r_L = \frac{E}{ZeB}$ and use this expression to derive the maximal energy that can be reached by a carbon atom ($Z = 6$) being accelerated in a galaxy with size $L = 30$ kpc and magnetic field $B = 1\ \mu$G.

Additional information provided during the exam.
1 kpc = $3.086\ 10^{19}$ m
elementary charge = 1.610^{-19} C
atomic mass unit = $1.7\ 10^{-27}$ kg
1 Gauss = 10^{-4} Tesla

(2) (Total 6 marks)
In a simplified model the development of the shower depends on the initial energy of the primary particle E_0, as well as the critical energy E_c, that is a property of the medium where the shower develops. For electromagnetic showers in air $E_c = 100$ MeV.

(a) (2 marks)
Compute the maximum number of particles in the shower generated by a photon of initial energy $E_0 = 1$ TeV, entering the atmosphere vertically.

(b) (1 mark)
How is the critical energy defined ?

(c) (1 mark)
For a vertical shower, the depth x in the upper atmosphere (in g/cm^2) can be expressed as a function of height h using the following expression:

$$x \propto X e^{-h/H}$$

where $X \simeq 1030$ g/cm^2, $H = 7$ km. Consider $X_0(air) = 36.6$ g/cm^2.
Calculate the maximum depth height h_{max} in the Heitler model.

(d) (2 marks)
Define the concept of the light pool, and compute the area of the light pool assuming the Cherenkov light to be emitted at height h_{max} with an aperture angle of 1.3 degrees.

(3) (Total 3 marks)
Why is the sky dark at night and not as bright as during daylight ? This question was asked by Heinrich Wilhelm Olbers (1758 - 1840), a German astronomer. He assumed that the cosmological principle is valid. He realized that the light intensity drops by r^{-2} and that at the same time the number of stars in a shell with thickness dr with a radius r increases as $4\pi r^2 dr$. Therefore, every portion of the sky must end up at a star, so there cannot be "darkness". This is known as the "Olbers paradox".

(a) (2 marks)
Discuss the solution to this paradox.

(b) (1 mark)
Try to help Olbers: might he have been right after all if he would have known about the cosmic microwave background radiation?

(4) (Total 7 marks)
Anti-protons constitute a sub-dominant component of cosmic rays, as they are mainly produced as a consequence of primary cosmic rays interactions with the interstellar medium (ISM). Cosmic-ray anti-proton production takes place via the following process:

$$p + p_{ISM} \rightarrow \bar{p} + p + p + p$$

(a) (1 mark)
Compute the 4-momentum squared in the laboratory frame for the initial state, assuming the target, p_{ISM}, at rest.

(b) (1 marks)
Compute the 4-momentum squared for the final state in the center of mass frame, as a function of the proton mass m_p, under the hypothesis that the particles are produced at rest.

(c) (2 marks)
Compute the threshold energy of the primary (high-energy) proton for this process, as a function of the proton mass m_p.

(d) (1 mark)
Compute the mean free path of the high-energy protons, assuming that the only relevant phenomena that affects their propagation in the galaxy is the interaction with

the ISM, whose density is n. The interaction cross section is given by $\sigma=$ 100 mb. Consider the following numerical values: $n =\ 1\ cm^{-3}$, 1 mb = 10^{-31} m^2.

(e) (2 marks)
Discuss the validity of the assumption made in sub-question (d), for the mean free path of protons in the galaxy. Discuss whether or not the same assumption is also valid for electrons.

(5) (Total 5 marks)
Neutrinos are elusive particles with no electric charge and very small mass compared to the charged leptons. They are created through the weak-interaction process.

(a) (1 mark)
What are the names of the force carriers for this process.

(b) (1 mark)
After traveling a distance L, a neutrino with an energy E has a certain probability to change its flavor state, e.g. from the muon to the tau flavor. In the atmosphere many muon neutrinos are created. Name the reaction(s) that describe the creation of the muon neutrinos in the atmosphere.

(c) (2 mark)
In a two-flavor scenario the appearance probability of a tau neutrino starting as a muon neutrino is given as:

$$P_{\nu_\mu \to \nu_\tau}(t) = | < \nu_\mu | \nu_\tau(t) > |^2 = \sin^2(2\theta_{23}) \sin^2[(E_2 - E_3)t/(2\hbar)]$$

where θ_{23} is the so-called mixing angle, and $E_{2,3}$ are the energies of the two mass eigenstates 2 and 3. Assuming that only the mass of these two eigenstates are different and not their momenta, proof that one can write the appearance probability as:

$$P_{\nu_\mu \to \nu_\tau}(t) = \sin^2(2\theta_{23}) \sin^2 \left[\frac{\Delta m^2 c^4}{4\hbar c} \frac{L}{E} \right]$$

where $\Delta m^2 c^4 = m_2^2 c^4 - m_3^2 c^4$.

(d) (1 mark)
Calculate the appearance probability of 1 GeV muon neutrino to be detected as a tau neutrino after passing through the Earth (diameter 13800 km). Use for the mixing angle $\theta_{23} = 45°$.
Use $\dfrac{1}{4\hbar c} = 1.27 \times 10^9$ eV^{-1}km^{-1} and $\Delta m^2 c^4 = 4.6 \times 10^{-5}$ eV^2.

(6) (Total 8 marks)
Charged particles loose energy when they travel through space.

(a) (2 marks)
Provide the important parameters for the energy loss mechanism in case these particles are traveling along magnetic field lines.

(b) (2 marks)
Describe the observational evidence that this mechanism is at work in the universe.

(c) (2 marks)
Provide the important parameters for the energy loss mechanism in case these particles are traveling through (intense) photon fields.

(d) (2 marks)
Describe the possible experimental evidence that this mechanism is at work in the universe.

1. Solutions

(1) **Question 1**

(a) If the energy gain per cycle is $\Delta E = \epsilon E_0$, where E_0 is the initial energy, the energy after one cycle will be $E_1 = E_0 + \Delta E = E_0(1+\epsilon)$, and after two cycles it will be $E_2 = E_1 + \Delta E = E_0(1+\epsilon)^2$. Following this process, the energy after n cycles is given by:

$$E_n = E_0(1+\epsilon)^n$$

(b) The fraction of remaining particles after n cycles is given by:

$$\frac{N_n}{N_0} = P_c^n$$

(c) The probability for a particle to be accelerated is given by the probability to stay for at least n cycles, where $n = \frac{lg(E_n/E_o)}{lg(1+\epsilon)}$. Substituting into the result from previous sub-question we get:

$$\frac{N_n}{N_0} = P_c^n = P_c^{\frac{lg(E_n/E_o)}{lg(1+\epsilon)}}$$

From previous formula we have:

$$lgN = n\ lgP_c = lg(\frac{E_n}{E_0})\frac{lgP_c}{lg(1+\epsilon)} + lgN_0$$

we can define the following quantity, that is independent on energy: $-s = \frac{lgP_c}{lg(1+\epsilon)}$, so we have ($E = E_n$):

$$lgN = -s\ lg(\frac{E}{E_0}) + lgN_0 \rightarrow N \propto (\frac{E}{E_0})^{-s}$$

Then the energy spectrum $\frac{dN}{dE}$ of the particles that have undergone the acceleration process is given by:

$$\frac{dN}{dE} \propto E^{-(s+1)} = \frac{dN}{dE} \propto E^{-\gamma}$$

where the spectral index $\gamma = s + 1$.

(d) The spectral index of the energy spectrum can be obtained from the previous formula, where we should add the information about the relative energy gain for the process, that so far we indicated as ϵ. Since it is stated in the text that we are in the framework of the first-order Fermi mechanism, this means that $\epsilon = \beta$. For the shock that we are considering we have $\beta = \frac{6\ 10^4 km/s}{3\ 10^5 km/s} \simeq 0.2$. The relative energy gain after one crossing will be:

$$E_1/E_0 = (1+\beta) = (1+0.2) = 1.2$$

so that the index is given by:

$$\gamma = \frac{-lgP_c}{lg(1+\beta)} + 1 = -lg(0.7)/lg(1.2) + 1 = 2.0 + 1 = 3.0$$

(e) For the particle of initial energy 10^6 eV to reach 10^{12}eV, the number of times it has to cross the shock front is given by:

$$E_n = E_0(1+\epsilon)^n \rightarrow 1TeV = 1MeV(1+\beta)^n \rightarrow n = \frac{lg(10^6)}{lg(1.2)} = 76$$

(f) The Larmor radius, also called gyroradius, is the radius of the circular path that charged particles follow when traversing a region with a uniform magnetic field.
We can first make the computation for a non-relativistic particle ($\gamma = 1$). Assuming that the velocity is perpendicular to the magnetic field of strength B, and that the particle charge is $Q = Ze$, at the equilibrium between the centrifugal and the Lorentz force we have:

$$ZevB = \frac{m\ v^2}{r} \rightarrow r_L = \frac{mv}{ZeB} = \frac{p}{ZeB}$$

For a relativistic particle the same formula holds (where $p = \gamma mv$).
In case of high energy relativistic particles $E >> mc^2$, so that $r_L = \frac{E}{ZeB}$ (use c=1).
We can use this expression to derive the maximal cosmic ray particle energy that be contained (remember to put the actual value of c to get the final numerical value, otherwise it has the wrong result with the wrong dimensions).
For a Carbon nucleus ($Ze = 6 \cdot 1.6\ 10^{-19}C$), accelerated in a region with size L =30 kpc= 9×10^{20} m and magnetic field B= 10^{-10} T, is $E_{max} = Ze\ c\ B\ L$.
The numerical result is:

$$E_{max} = 26\ J = 1.6 \cdot 10^{20} eV$$

(2) **Question 2**

(a) The maximum number of particles is given by:

$$N_{max} = \frac{E_0}{E_c} = 2^{x_{max}} = \frac{10^{12}eV}{10^8 eV} = 10^4$$

(b) In order to compute the h_{max} we first have to compute the depth of the shower maximum from the previous formula. In units of radiation lengths, it is given by:

$$x_{MAX} = ln(\frac{E}{E_0})/ln2 = \frac{ln10^4}{ln2} X_0 = 13.3 X_0$$

To get the final value in g/cm^2 we have to use the value given for the radiation lenght in air:

$$x_{MAX} = 13.3\ X_0 = 13.3 \cdot 36.6 g/cm^2 = 487\ g/cm^2$$

then using the relation between depth and height we find that:

$$x_{MAX} \propto X exp(-h_{MAX}/H) \rightarrow h_{MAX} = -H\ ln(\frac{x_{MAX}}{1032 g/cm^2}) \simeq 5km$$

(c) In the Heitler model it is assumed that the cascade multiplication will go on until the particle energy reaches the value $E = E_c$, the critical energy, when we suppose that ionization loss becomes dominant and that no further radiation or pair production processes are possible.

(d) The light pool is the area illuminated within the light cone produced by the shower initiated by the primary gamma ray in the atmosphere. The light pool can be derived by a opening angle θ_C from the shower maximum. If we call θ_C the Cherenkov angle we have that

$$r_p = h_{MAX} tan\ \theta_C\ =\ 5\ km\ tan(1.3deg)\ =\ 113m$$

then the light pool area is given by:

$$A = \pi r_p^2 \simeq 4\ 10^4\ m^2$$

(3) **Question 3: Olbers paradox**

(a) There are several reasons why Olbers arguments are invalid.
Each correct argument provides 0,5 mark.

- First, we believe the observable universe is not infinite but has a finite age, and began at a time t_0 in the past with the Big Bang which started off the Hubble expansion. This means that light can only reach us from a maximum horizon distance ct_0, and the flux must be finite.
- A second point is that the light sources (stars) are finite in size, so that nearby sources will block out light from more distant sources. Their light is absorbed exponentially with distance by the intervening stars and dust.
- Third, stars only emit light for a finite time t, and the flux from the most distant stars will therefore be reduced by a factor t/t_0.
- Finally, the expansion of the universe results in an attenuation at large enough redshifts of light of any particular frequency; for example, red light will disappear into the infrared and the flow of light energy will fall off.

(b) However, we may remark tha the 2.7 K microwave background radiation which is the cooled and red-shifted remnant of the original expanding fireball of the Big Bang, although invisible to the eye, is just as intense at night time as during the day. So in this sense Olbers was right, be it that the light was not created by stars, but by the relics from the Big Bang.

(4) **Question 4: cosmic ray antiprotons**

(a) Considering the ISM at rest, then $E_{ISM} = m_p$. The four momentum squared in the laboratory frame ($P^2_{LAB} = S_{LAB}$) is thus given by:

$$S_{LAB} = (E_p + E_{ISM})^2 - p^2 = E_p^2 + m_p^2 + 2E_p m_p - p^2 = 2m_p^2 + 2m_p E_p$$

(b) In the center of mass, if we consider that the particles are produced at rest, we have $S_{CDM} = (4m_p)^2$.

(c) The energy threshold is obtained by requiring that:

$$2m_p(E_p + m_p) \geq (4m_p)^2 \rightarrow E_p \geq 7m_p$$

The minimum energy that the proton should have in order to produce the 4 particles at rest in the center of mass is thus $E_p = 7m_p$.

(d) The mean free path for particles traveling in the ISM is given by: $\lambda = \frac{1}{n_{ISM}\sigma_{ISM}}$. Substituting the numerical values we have:

$$\lambda = \frac{1}{1\ cm^{-3} 100 \cdot 10^{-27} cm^2} = \frac{1}{10^{-25} cm^{-1}} = 10^{25}\ cm\ \ =\ \ 3,2\ Mpc$$

(e) If we consider galactic particles, i.e. with energies below $10^{15} eV$, propagating in the ISM will encounter radiation fields and other particles. The interaction cross section is proportional to the inverse of the square of the mass m^{-2} such that the interaction probability for electrons is roughly 10^6 times higher than for protons. The assumption is reasonable for protons, but not for electrons. As an order of magnitude, it is considered that more than 90% of electrons with GeV-TeV energy travel only few kpc. For much higher energies, where we talk about extra-galactic cosmic rays, the GZK effect should also be taken into account, since the interaction with Cosmic Microwave Background radiation also limits the horizon of cosmic rays.

(5) **Neutrino oscillations**

(a) the charged W^+ and W^- bosons and the neutral Z boson (in case of neutrino-pair production).

(b) nucleon-nucleon collisions or nucleus-nucleon collisions lead to the production of charged pions, which decay to leptons, partly as muon neutrinos. By the way, a neutrino oscillation is not a reaction, but an effect of quantum mechanics (probability to find a particle in a particular eigen state).

(c) We assume that the mass of the neutrinos is small compared to their energy ($v = c$). And we assume that the momenta of the two eigenstates are equal ($p_2 = p_3 = p$). With these two assumptions, we can write:

$$E_i^2 = p^2c^2 + m_i^2c^4$$

$$E_i = pc\sqrt{1 + \frac{m_i^2c^4}{p^2c^2}} \approx pc\left[1 + \frac{1}{2}\frac{m_i^2c^4}{p^2c^2}\right] = pc + \frac{1}{2}\frac{m_i^2c^4}{pc}$$

Therefore, the energy difference will be:

$$E_2 - E_3 = \frac{m_2^2c^4 - m_3^2c^4}{2pc} = \frac{\Delta m^2c^4}{2pc}$$

Furthermore,

$$\frac{t}{\hbar} = L\frac{1}{v}\frac{1}{\hbar} = L\frac{1}{c\hbar} = L\frac{p}{E\hbar}$$

Therefore, the argument of the sine function will become:

$$\frac{(E_2 - E_3)t}{(2\hbar)} = \frac{\Delta m^2c^4}{2pc}\frac{Lp}{E\hbar} = \frac{\Delta m^2c^4}{4}\frac{L}{c\hbar E}$$

where we use $E = pc$.

(d) We need to calculate:

$$\begin{aligned}\frac{\Delta m^2}{4\hbar c}\frac{L}{E} &= \left[4.6 \times 10^{-5}\right] \times \left[1.27 \times 10^9\right]\frac{13800}{1 \times 10^9} \\ &= 1.27 \times 4.6 \times 10^{-5} \times 13800 = 0.806 \text{ rad} \\ &= 46.2^\circ\end{aligned}$$

And $\sin^2(46.2^\circ) = 0.52$.
There is 52% chance the muon neutrino will appear als a tau neutrino.

(6) **Energy losses in the universe**

(a) Energy loss through acceleration / gyration for charged particles following magnetic field lines. So-called synchrotron radiation. Parameters are energy, mass, and charge of the particles, strength of the magnetic field.

(b) Measurements from radio up to X-rays provide information that synchrotron radiation occurs in the universe.

(c) The passage of particles through intense photon fields may lead to pion emission (GZK effect). Important parameters, energy of the particles, energy of the photons, probability to excite delta resonance or to breakup a heavy nucleus into fragments by Coulumb dissociation. Another effect is the creation of an electron-positron pair by the $p + \gamma$ reaction (aka Bethe Heitler process). By the way, in Compton scattering particles gain energy, through the interaction of high-energy photon with a low energy electron. In the **inverse** Compton scattering process, particles will indeed loose energy.

(d) Cosmic ray spectrum beyond 10^{19} eV provides a piece of evidence; still needs to be confirmed. The pair creation process could explain the dip in the cosmic ray spectrum between the knee and the ankle around 10^{19} eV.